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by

Richard B. Barrar

ABSTRACT

Periodic solutions of the equations of motion of a satellite of an oblate planet are investigated, using methods developed by Poincaré (1892). The treatment covers both critical and non-critical angles.

INTRODUCTION

Much work has recently been devoted to the motion of a satellite of an oblate planet; see Brouwer (1959), Garfinkel (1959), Kozai (1962). However, very little attention has been paid to the existence of periodic orbits. McMillan (1910) discussed periodic orbits about an oblate planet. However, all his orbits reduce to circular orbits in the unperturbed case. In the present paper, we apply methods developed by Poincaré (1892) to investigate periodic orbits. We cover both critical and non-critical angles. It is found that the periods are completely different at critical and non-critical angles.

Background

In order to make the presentation as clear as possible, we will treat only the case when the planet's potential is of the form:

(1) $$V = (\mu/r)(1-2k_2P_2(\sin\beta)/r^2) .$$

It will be clear to the reader that everything said will also apply to the more general potential:

$$V = (\mu/r)\,[1 + \sum_{p=2}^{\infty} B_p P_p(\sin\beta)/r^p] \tag{2}$$

In equations (1) and (2), I have used the notation of Brouwer (1959), namely, μ, k_2, and B_p are physical constants. β is the latitude, r the distance from the center of the planet. $P_n(\sin\beta)$ are Legendre polynomials.

In terms of the Delaunay variables, the Hamiltonian corresponding to the potential (1) can be written (see Brouwer, 1959):

$$F = F_o(x_1) + k_2\,F_{11}(x_1,x_2) + k_2\,F_{12}(x_1,x_2,y_1,y_2) \tag{3}$$

with

$$F_o = (1/2)(\mu/x_1)^2 \tag{4}$$

$$F_{11} = (-\mu/2)(1-3(x_3/x_2)^2)(\mu/x_1x_2)^3 \tag{5}$$

$$F_{12} = \sum_{j>0} P_j(x_1,x_2,x_3)\cos jy_1 + \sum_{j\neq 0} Q_j(x_1,x_2,x_3)\cos(jy_1 + 2y_2) \tag{6}$$

where

$x_1 = (\mu a)^{\frac{1}{2}}$ — a = semi-major axis

$x_2 = x_1(1-e^2)^{\frac{1}{2}}$ — e = eccentricity

$x_3 = x_2 \cos I$ — I = inclination

y_1 = mean anomaly

y_2 = argument of the pericenter

y_3 = longitude of the ascending node

($x_1, x_2, x_3, y_1, y_2, y_3$ are canonical variables in the Hamiltonian (3)). See Poincaré § 8.

x_3 is a constant in the above formulas since the conjugate momenta y_3 is not present. After x_1, x_2, y_1, y_2 have been found in the above Hamiltonian, one can solve for y_3 by the formula:

$$y_3 = y_3^o - \int_o^t \frac{\partial F}{\partial x_3}(x_1, x_2, x_3, y_1, y_2)\,dt.^* \tag{7}$$

Since (7) is merely an integration, we will restrict ourselves to solving the problem (3) with two degrees of freedom. Thus, when we speak of periodic solutions, we mean with respect to the four variables x_1, x_2, y_1, y_2. It will be clear to the reader that our methods would also carry over with the addition of x_3 and y_3; but, once again, to increase the clarity of presentation this added complication will not be introduced. Now define

$$n_1^o = \partial F_o(x_1^o)/\partial x_1 \tag{8}$$

---

*Any quantity throughout this paper with a superscript o will mean it is a constant; thus y_3^o is a constant.

$$n_j^1 = - \partial F_{11}(x_i^o)/\partial x_j$$

If $n_2^1 = 0$, we have the so-called critical angle case. If $n_2^1 \neq 0$, we have the non-critical angle case.

Existence of Periodic Orbits in General

In this section we will briefly review the results of Poincaré (1892) § 42 on the existence of periodic solutions. We will show that these results are not directly applicable to the Hamiltonian arising from the motion of a satellite of an oblate planet. In latter sections we will show how the Hamiltonian can be transformed so that the Poincaré criteria are applicable.

Poincaré (1892) § 42 considers a Hamiltonian of the form:

$$H = H_o(a_1, a_2) + \sum_{i=1}^{\infty} \epsilon^i H_i(a_1, a_2, w_1, w_2) \tag{9}$$

where a_1, a_2, w_1, w_2 are canonical variables, corresponding to our previous x's and y's with

$$H_i = \Sigma\, B(a_1, a_2) \cos\, (m_1 w_1 + m_2 w_2 + h(a_1, a_2)) \tag{10}$$

and where the summation is over all integers m_1, m_2.

If we set $\epsilon=0$ in (9) and solve the resulting equations of motion, we obtain the unperturbed solution:

$$a_i = a_i^o \tag{11a}$$

(11b) $$w_i = \overline{n}_i^o\, t + d_i^o$$

(11c) $$\overline{n}_j^o = -\,\partial H_o(a_i^o)/\partial a_j$$

Poincaré further assumes the a_i^o have been so chosen that $(\overline{n}_2^o/\overline{n}_1^o) = (p/q)$ with p and q integers. Then considering w_1, w_2 angle-like variables, Poincaré calls the unperturbed solution periodic of period $T = 2\pi q/\overline{n}_1^o = 2\pi p/\overline{n}_2^o$. p may be zero, then $T = 2\pi/\overline{n}_1^o$.

The principal problem is to find periodic solutions of the same period T, for small values of ϵ, that will approach the unperturbed solution as $\epsilon \to o$. Mathematically this can be formulated as follows. Let

(12a) $$a_i(0) = a_i^o + b_i(\epsilon); \quad w_i(0) = d_i^o + c_i(\epsilon), \text{ and}$$

(12b) $$a_i(T) = a_i^o + b_i(\epsilon) + \Psi_i; \quad w_i(T) = d_i^o + c_i(\epsilon) + \overline{n}_i^o\, T + \phi_i\ .$$

To insure periodicity and the desired behavior as $\epsilon \to o$, one must determine the a_i^o, d_i^o, so that $\Psi_i = \phi_i = o$ for all sufficiently small ϵ, and such that $b_i(\epsilon)$, $c_i(\epsilon)$ approach zero with ϵ. (These are exactly the same equations that Lefschetz (1957) p. 163 considers.)

We first point out that since H = constant is an integral with $(\partial H/\partial a_1) \neq 0$, it follows that if

(13) $$\Psi_2(b_1,b_2,c_1,\epsilon) = \phi_1(b_1,b_2,c_1,\epsilon) = \phi_2(b_1,b_2,c_1,\epsilon) = 0$$

then Ψ_1 will also be zero (see Siegel (1956) top of page 126 or Poincaré,

Vol. I, p. 87). Hence, we need solve only (13). Moreover by a change in t, we may assume $d_1^o = c_1(\epsilon) = 0$.

For ϕ_1 one finds

$$\phi_1 = \int_o^T ((dw_1/dt) - \bar{n}_1^o)dt = \int_o^T ((-\partial H/\partial a_1) - \bar{n}_1^o)dt \tag{14}$$

$$= T((\partial H_o(a_i^o + b_i)/\partial a_1) - \bar{n}_1^o) + O(\epsilon)$$

similarly for ϕ_2

$$\phi_2 = T((\partial H_o(a_i^o + b_i)/\partial a_2) - \bar{n}_2^o) + O(\epsilon) \tag{15}$$

For ψ_2 one finds

$$\psi_2 = \int_c^T (da_2/dt)dt = \int_o^T (\partial H/\partial w_2) \tag{16}$$

$$= \sum \epsilon^i \int_o^T (\partial H_i/\partial w_2)dt = \epsilon(\partial [H_1]/\partial d_2^o) + O(\epsilon^2)$$

where if H_1 is defined in (10) then

$$[H_1] = T\ \Sigma' B(a_1^o, a_2^o)\cos(m_1 d_1^o + m_2 d_2^o + h(a_1^o, a_2^o)) \tag{17}$$

and where Σ' means only sum over the terms where

$$m_1 \bar{n}_1^o + m_2 \bar{n}_2^o = 0\ . \tag{17a}$$

(This follows from (16) on substituting the unperturbed solution in H_1. Note

also that we assume $d_1^o = 0$.)

It now follows from the implicit function theorem, that if one is to solve the three equations (13) for small values of ϵ, the following conditions are sufficient:

$$\left(\frac{\partial^2 H_o}{\partial a_1 \partial a_1} \cdot \frac{\partial^2 H_o}{\partial a_2 \partial a_2} - \left(\frac{\partial^2 H_o}{\partial a_1 \partial a_2}\right)^2\right)\Bigg|_{a_i = a_i^o} \neq 0 \tag{18}$$

and $d_2^o = \overline{d}_2^o$ is so chosen that:

$$\partial[H_1(a_1^o, a_2^o, \overline{d}_2^o)]/\partial d_2^o = 0 \tag{19a}$$

$$\partial^2[H_1(a_1^o, a_2^o, \overline{d}_2^o)]/\partial d_2^o \partial d_2^o \neq 0 \tag{19b}$$

If the Poincaré criteria (18) and (19) are fulfilled, we are guaranteed periodic solutions for sufficiently small values of ϵ. If $[H_1] \not\equiv$ constant, since $[H_1]$ is periodic it will always have a maximum and minimum; thus there will always exist a $\overline{d}_2^o$, satisfying (19a), (19b).

Let us now see where the Hamiltonian (3) fails to fit these criteria. The obvious choice for H_o, H_1 is (using the definitions (3), (4), (5), (8)) (we clearly equate x_i with a_i, and y_i with w_i):

$$H_o = F_o + k_2 F_{11}; \qquad \epsilon H_1 = \epsilon F_{12} \tag{20}$$

where in (20) we consider k_2 as a fixed quantity, ϵ as variable. Thus (see (11c))

(21) $$\bar{n}_1^o = n_1^o + k_2 n_1^1; \quad \bar{n}_2^o = k_2 n_2^1$$

With this definition of H_o, it follows that the development in the present section is applicable. Furthermore, it follows that for most values of a_1^o, a_2^o, a_3^o (18) is fulfilled; for those values where (18) is not fulfilled one can treat the square of the Hamiltonian as suggested by Poincaré § 43 to obtain a new Hamiltonian where (18) is fulfilled. Hence (18) causes no trouble.

It is (19b) that is not fulfilled. From (6) it follows that m_2 is either 0 or 2; from (21) it follows that $\bar{n}_2^o$ is a very small number. Hence there are no terms in H_1, where (17a) is fulfilled; thus $[H_1] \equiv 0$, and (19b) is not fulfilled.

In the next section we will introduce a transformation that will convert the Hamiltonian (3) to one where the criteria (18), (19) are fulfilled.

The Poincaré--von-Zeipel Transformation

It is clear that if $[F_1] \equiv 0$, perhaps if we develop in powers of ϵ we will eventually come to a term corresponding to $[F_1]$ that will not be identically zero. The most straightforward way to do this is by what is now known as the von-Zeipel transformation. However, it should be mentioned that practically the whole of Poincaré (1893) Volume II is devoted to this same type of transformation. We follow principally Poincaré §134 and §125 in our development.

Let

(22) $$\Sigma_p = \sum_{j=o}^{p} \epsilon^j S_j(a_1, a_2, y_1, y_2)$$

with

$$S_o = a_1y_1 + a_2y_2 \tag{23}$$

Now consider the partial differential equation:

$$F(\partial\Sigma_p/\partial y_1, \partial\Sigma_p/\partial y_2, y_1, y_2) = C_o(a_1) + \epsilon C_1(a_1,a_2) \cdots + \epsilon^p C_p(a_1,a_2) \tag{24}$$

where F is the Hamiltonian (3) (with k_2 changed to ϵ), and the $C_i(a_1,a_2)$ are to be determined.

If like powers of ϵ are equated in (24), one finds the series of equations

$$F_o(a_1) = C_o(a_1) \tag{25}$$

$$n_1^o(\partial S_j/\partial y_1) = n_2^1(\partial S_{j-1}/\partial y_2) + q_2^1(\partial S_{j-1}/\partial y_2) + \Phi_j - C_j(a_1,a_2) \tag{26}$$

with Φ_j a polynomial of the j-th degree in $(\partial S_k/\partial y_i)$ $k < j-1$, the coefficients of this polynomial being either $\partial^n F_o(a_1)/\partial x_1^n$ or $\partial^p F_1(a_1,a_2,y_1,y_2)/\partial^q x_1 \partial^{p-q} x_2$, and

$$\begin{aligned} n_1^o &= \partial F_o(a_1)/\partial x_1 \\ n_2^1 &= \partial F_{11}(a_i)/\partial x_2 \\ q_2^1 &= \partial F_{12}(a_i,y_i)/\partial x_2 \end{aligned} \tag{27}$$

To any solution $S_j^{(1)}$ of (26), we can always add on a solution $S_j^{(2)}$ that only depends on y_2.

Let us first proceed formally and assume at some stage we have computed $S_o, \ldots, S_{j-2}$, and $S_j^{(1)}$. We now use equation (26) to compute $S_{j-1}^{(2)}$, and $S_j^{(1)}$. Since Φ_j is assumed known we have:

$$
\text{(28)} \quad
\begin{aligned}
q_2^1(\partial S_{j-1}^{(1)}/\partial y_2) + \Phi_j = & \sum_{m_1 \neq o} A(a_1,a_2) \cos (m_1 y_1 + m_2 y_2 + h(a_1,a_2)) \\
& + \sum_{m_2 \neq o} A'(a_1,a_2) \cos (m_2 y_2 + h(a_1,a_2)) \\
& + C'(a_1,a_2)
\end{aligned}
$$

Now determine $C_j(a_1,a_2)$ and $S_{j-1}^{(2)}$ to eliminate the last two terms on the right-hand side; thus

$$
\text{(29)} \quad C_j(a_1,a_2) = C'(a_1,a_2)
$$

$$
\text{(30)} \quad S_j^{(2)} = -\sum_{m_2 \neq o} (A'(a_1,a_2)/m_2 n_2^1) \sin (m_2 y_2 + h(a_1,a_2))
$$

If $S_{j-1} = S_{j-1}^{(1)} + S_{j-2}^{(2)}$ is used in (28) rather than $S_{j-1}^{(1)}$, then since q_2^1 has only terms which depend on y_1, only the first series on the right-hand side is affected. Thus with (28), (29), (30) inserted in (26) we may choose a solution $S_j^{(1)}$ of the form:

$$
\text{(31)} \quad S_j^{(1)} = -\sum_{m_1 \neq o} (A(a_1,a_2)/m_1 n_1^o) \sin (m_1 y_1 + m_2 y_2 + h(a_1,a_2))
$$

With this method of iteration we determine $S_1, \ldots, S_p$. Away from the critical angle both n_1^o, n_2^1 are $O(1)$; hence there are no small divisors in (30), (31).

The above formal method of proceeding can be fully justified. First we note that the sum in (30) contains only a finite number of terms; hence term by term integration is permissible.

The justification of term-by-term integration in (31) is slightly more involved. First we recall that essentially $F_1 = (a/r)^3 + (a/r)^3 \cos 2(f+y_2)$, and then the following formulas (see Brouwer 1959, Wintner 1947):

$$(a/r) = 1+2 \sum_{n=1}^{\infty} J_n(ne) \cos ny_1$$

$$\cos f = -e + (1-e^2) \sum_{n=-\infty}^{\infty}{}' J_{n-1}(ne) \cos ny_1$$

$$\sin f = (1-e^2)^{1/2} \sum_{n=-\infty}^{\infty}{}' J_{n-1}(ne) \sin ny_1$$

$$\partial\psi/\partial x_1 = (1/ex_1)(x_2/x_1)^2(\partial\psi/\partial e)$$

$$\partial\psi/\partial x_2 = -(1/ex_1)(x_2/x_1)(\partial\psi/\partial e)$$

$$\partial(a/r)/\partial e = (a/r)^2 \cos f$$

$$\partial f/\partial e = ((a/r) + (x_1/x_2)^2) \sin f$$

From these formulas, together with the estimates for $J_m(mz)$ given in Wintner 1947 §294, plus standard theorems on the multiplication and integration of Fourier series, it follows that our formal procedure is justified.

With Σ_p, $C_j(a_1,a_2)$, $S_j=S_j^{(1)} + S_j^{(2)}$ defined as above, we follow Poincaré in changing to new canonical variables a_i,w_i by the formulas:

(32)
$$x_i = (\partial\Sigma_p/\partial y_i)$$
$$w_i = (\partial\Sigma_p/\partial a_i)$$

The Hamiltonian becomes:

(33)
$$H = F(x_i,y_i) = F(\partial\Sigma_p/\partial y_i)y_i) = C_o+\epsilon C_1+\ldots+\epsilon^{o}C_p+\epsilon^{p+1}\Phi_{p+1}(a_1,a_2,y_1,y_2,\epsilon)$$

where clearly Φ_{p+1} is periodic of period 2π with respect to y_1,y_2 and expandable as a power series in ϵ.

Write the change of variable equations as:

(34)
$$x_i = a_i + \partial(\Sigma_p-S_o)/\partial y_i \qquad y_i = w_i - \partial(\Sigma_p-S_o)/\partial a_i$$

Because the S_j are periodic with respect to y_1,y_2, it follows from the above equations that if y_i is changed to $y_i + 2k_i\pi$ and w_i to $w_i + 2k_i\pi$, the equations will not change. Hence x_i and y_i-w_i are periodic of period 2π with respect to w_1,w_2. The Hamiltonian (33) therefore is also periodic of period 2π with respect to w_1 and w_2. Thus the Poincaré--von-Zeipel transformation has the very important and interesting property that it takes periodic solutions in a_i,w_i into periodic solutions in x_i,y_i and vice versa. Hence, to study periodic solutions it does not matter which set of variables we use.

In summary: x_j and y_j are functions of a_i,w_i, being periodic of period 2π with respect to w_1 and w_2. The equations of motion with respect to the canonical variables a_i,w_i comes from the Hamiltonian:

$$(35) \qquad H = H_o(a_1,a_2) + \sum_{i=p+1}^{\infty} \epsilon^i H_i(a_1,a_2,w_1,w_2)$$

with $H_o(a_1,a_2) = F_o(a_1) + \sum_{i=1}^{p} k_2^i \; C_i(a_1,a_2)$, and H_i of the form (10).

Periodic Orbits for Non-critical Angles

We note that we started with an $F_1 \doteq H_1$ of the form (10), where m_2 had only the values 0 and 2. Then if we defined $[F_1] = [H_1]$ as in (17), $[F_1] \equiv 0$. However, in the construction of the Poincaré--von-Zeipel transformation, the series $(\partial S_i/\partial y_j)$ are multiplied with each other and in the process the range of values of m_2 grows. Thus, we may assume that for non-critical angles, we have transformed to (35) and that we have chosen p so large that $[H_{p+1}] \neq 0$. In fact, it is clear from our method of construction that we may assume $[H_{p+1}] = A(a_1,a_2) \cos ((p+1)d_2^o + h(a_1,a_2))$. Hence, the Poincaré theory of periodic orbits is now applicable to non-critical angles orbits, with $\bar{n}_j^o$ defined by (35) and (11c).

Hence, we conclude that if we assume the initial eccentricity is not zero, (so that the P_j,Q_j in (6) does not vanish) and the initial inclination angle is not the critical angle, then for any initial value of semi-major axis, eccentricity, inclination and for certain values of the initial value of perigee (those values satisfying the Poincaré criteria (19)), the equations of motion of a satellite of an oblate planet will have periodic solutions. Note that the period for non-critical inclinations depends on the period it takes the perigee to make one revolution.

Furthermore, since the equation giving rise to the initial value of

perigee comes from a term of $O(\epsilon^{p+1})$ where p is very large, it is conjectured that by changing the oblate planet's potential very slightly it would be possible to assign the initial value of perigee arbitrarily and still obtain a periodic solution.

Periodic Orbits at the Critical Angle

At the critical angle $n_2^1 = 0$. However, irrespective of the value of n_2^1, if we apply the Poincaré--von-Zeipel transformation, we obtain the following equation for $S_1^{(1)}$.

$$n_1^o(\partial S_1^{(1)}/\partial y_1) = F_1(a_1,a_2,y_1,y_2) - C_1(a_1,a_2)$$

Because F_1 contains no terms of the form $\cos m_2y_2$, this equation is easily solved (see Brouwer (1959) p. 380-381). Now letting $\Sigma_p = S_o + S_1^{(1)}$, the transformed Hamiltonian is of the form (see Brouwer (1959) p. 385):

$$H = F_o(a_1) + \epsilon F_{11}(a_1,a_2) + \epsilon^2[A_o(a_1,a_2) + A_1(a_1,a_2)\cos 2w_2 + \sum_{m_1\neq o} A_2(a_1,a_2)\cos(m_1w_1 + m_2w_2)] + O(\epsilon^3)$$

It is possible to apply the results of Poincaré §42 directly to this Hamiltonian. The unperturbed Hamiltonian is

$$H_o = F_o(a_1) + k_2\, F_{11}(a_1,a_2)$$

with

$$\bar{n}_1^o = -(\partial F_o(a_1^o)/\partial a_1) - k_2 \, (\partial F_{11}(a_1^o, a_2^o)/\partial a_1)$$

$$\bar{n}_2^o = -k_2 \, \partial F_{11}(a_1^o, a_2^o)/\partial a_2 = 0$$

thus the unperturbed orbit has period $T = 2\pi/n_1^o$ (there is nothing in Poincaré §42 that does not permit $\bar{n}_2^o = 0$; in fact, in §44 he reduces the general case to this situation).

Now $[H_2] = A_o(a_1, a_2) + A_1(a_1, a_2) \cos 2w_2$ and thus by the Poincaré criteria (18), (19) it follows that at the critical inclination there are periodic orbits of period $T = 2\pi/\bar{n}_1^o$ for all semi-major axis values, and all eccentricities not identically zero, and for initial values of perigee either 0 or π.

(In a private communication W. T. Kyner states that he has also obtained this result.)

Conclusion

The existence of periodic orbits at both critical and non-critical angles has been shown for all semi-major axis and eccentricity values and for the positions of perigee fulfilling (19). The frequency of the periodic orbit at the critical angle is the usual anomalistic frequency. The frequency of the periodic orbit at non-critical angles is the frequency of the motion of perigee (draconic frequency).

An interesting problem that remains to be discussed is how the periodic orbits away from the critical angle will approach the periodic orbits at the critical angle as the inclination changes.

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